Making Meaning from Multiple Strategies
NCTM K-2 Workshop
University of Illinois at Chicago

http://www.lsri.uic.edu/
Making Meaning from Multiple Strategies
K-2 Workshop

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Teaching Integrated Mathematics and Science (TIMS) Project
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How many ways can you solve this problem?

Chris’s group made 28 hats. Julia’s group made 44 hats. How many hats did both groups make altogether?
I can think about it better if I make a number line in my head. I think about starting at 44, moving forward 30 and then back 2, since 28 is 2 less than 30. I can write it like this.

Chris's Strategy:

I start at 44 and then add on 30, going by tens: 54, 64, 74. Subtract 2 and it is 72. 72 hats.
Julia's Strategy:

 Altogether we made 72 hats. I broke the numbers into tens and ones: 20 + 40 is 60, 8 and 4 is 12, 60 + 12 is 72. We made 72 hats.

\[
\begin{align*}
28 & = 20 + 8 \\
+ 44 & = 40 + 4 \\
60 + 12 & = 72 \text{ hats}
\end{align*}
\]
What do you notice about this student?
Goals for this Session

• Briefly establish a rationale for helping students develop multiple and flexible computation strategies.

• Analyze instructional strategies that encourage students to develop meaning and proficiency with those computation strategies.
Math Trailblazers
Research and Revision Study

2006–2009  Revision and field test of new materials in grades 1–5
2008–2009  Student Achievement Study
2010–2014  Final revision of materials for publication
Research Studies

- Field Test Study–UIC – 2006–2010
- Student Achievement Study–UIC – 2009–2011
- Embedded Assessment Study–UIC – 2010 - 2014
Research Studies

- Whole Number Study—UIC & KSU 2003–2008
- Implementation Study—UIC 2003–2006
- Fractions and Ratios—UMN 2004–2006
- Video Study—UIC 2003–2006
- Field Test Study—UIC 2006–2010
- Student Achievement Study—UIC 2009–2011
- Embedded Assessment Study—UIC 2010 - 2014
Why do we need a range of strategies for whole number computation?

Rationale #1

A range of strategies allows for sense-making and development of conceptual understanding.
The Iceberg Model

Adapted from: Webb, Boswinkle, & Dekker, 2008
“To find one’s way around the mathematical terrain, it is important to see how the various representations connect with each other, how they are similar, and how they are different. The degree of students’ conceptual understanding is related to the richness and extent of the connections they have made.”

- National Research Council, 2001
Stages of Conceptual Development

1. Direct Modeling
2. Counting Strategies
3. Reasoning from Known Facts

Carpenter, 1999; National Research Council, 2001
Stages of Conceptual Development

Direct Modeling

Counting Strategies

Reasoning from Known Facts

Counting All

\[
\begin{align*}
\bullet\bullet\bullet\bullet\bullet\bullet &\quad \bullet\bullet\bullet \\
1 &\quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\end{align*}
\]

\[5 + 3 = 8\]

Counting On

\[
\begin{align*}
\bullet\bullet\bullet\bullet\bullet\bullet &\quad \bullet\bullet\bullet \\
5 &\quad 6 \quad 7 \quad 8 \\
\end{align*}
\]

\[5 + 3 = 8\]

Reasoning from Known Facts

\[
\begin{align*}
\bullet\bullet\bullet\bullet\bullet\bullet &\quad \bullet\bullet\bullet \\
9 &\quad 6 = 10 \quad 5 = 15 \\
\end{align*}
\]

Carpenter, 1999; National Research Council, 2001
Stages of Conceptual Development

Direct Modeling

Counting Strategies

Reasoning from Known Facts

Carpenter, 1999; National Research Council, 2001
Why do we need a range of strategies for whole number computation?

Rationale #2

A range of strategies promotes computational fluency.

- NRC, 2001
What strategy would you use?

Would you use the same strategy?

$$9 + 2$$
What strategy would you use?

104 – 89 =

Is the strategy efficient?
Why do we need a range strategies for whole number computation?

Rationale #3

A range of strategies helps students *access and respond to* mathematical contexts.
There are 5204 Chocos. A customer came in and bought 565. Another customer came in and wanted to buy 4859 pieces of candy. Was there enough candy in the store so that he could buy that much?
Why do we need a range of strategies for whole number computation?

Rationale #4

A range of strategies supports a range of student *identities* and *needs*. 
Stages of Conceptual Development

Direct Modeling

Counting Strategies

Reasoning from Known Facts

Estimation  Mental Math

Paper and Pencil

Carpenter, 1999; National Research Council, 2001
Design Goals

• Make room for Invention
• Attend to representation and strategy
• Explicitly make connections
• Organize strategies and inventions
• Attend to strategy choice
Stages of Conceptual Development

- Direct Modeling
- Counting Strategies
- Reasoning from Known Facts

Carpenter, 1999; National Research Council, 2001

What stages are supported by this activity?
Design Goals

• Make room for Invention
• Attend to representation and strategy
• Explicitly make connections
• Organize strategies and inventions
• Attend to strategy choice
“Discussions that focus on cognitively challenging mathematical tasks, namely those that promote thinking, reasoning, and problems solving, are a primary mechanism for promoting conceptual understanding of mathematics.”

Smith, Hughes, Engle & Stein, 2009, p. 549
Promote Problem Solving and Invention

"100 Link chain"

Segment 2
Stages of Conceptual Development

Direct Modeling  Counting Strategies  Reasoning from Known Facts

What is stages are supported by this activity?

Carpenter, 1999; National Research Council, 2001
Make Room for Invention
Stages of Conceptual Development

Direct Modeling

Counting Strategies

Reasoning from Known Facts

What stages are supported by this activity?

Carpenter, 1999; National Research Council, 2001
How many all together?
Design Goals

• Make room for Invention
• **Attend to representation and strategy**
• Explicitly make connections
• Organize strategies and inventions
• Attend to strategy choice
Attend to representation and strategy

Show how you solve each problem. Use connecting cubes, ten frames, or number lines.

1. John has 5 red balloons and 3 blue balloons. How many balloons does he have?
Attend to representation and strategy
Attend to representation and strategy

How can you use one fact to solve another?

4 - 4  4 - 3  10 - 7  9 - 7
# Attend to Representation and Strategy

<table>
<thead>
<tr>
<th>One Strategy</th>
<th>Another Strategy</th>
</tr>
</thead>
</table>
| 1. **A.** $8 + 9$  
Start by adding $8 + 8$. | **B.** $8 + 9$  
Start by splitting 8 into $7 + 1$. |
5 + 6

\[
\begin{array}{c}
\cdot \cdot \cdot \cdot \\
\times \times \times \times \\
\hline \\
5 & 5 \\
\hline \\
10 & 11 \\
\end{array}
\]

8 + 7

\[
\begin{array}{c}
\cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot \\
\end{array}
\]

8 + 8 = 16 because
\[
8 + 2 + 6 = 16
\]
so 8 + 7 = 15 because
\[
8 + 2 + 5 = 15
\]

8 + 9

8 + 8 = 16
so 8 + 9 = 17 or one more

9 + 6

10 + 6 = 16 so 9 + 6 is one less or 15

\[
\begin{array}{c}
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\end{array}
\]

+10

+6

9 + 5

10 + 5 = 15
so
9 + 5 = 14

Addition Strategies

5 + 3 = 8

\[
\begin{array}{c}
1 & 2 & 3 & 4 \hline \\
5 & 6 & 7 & 8 \\
\end{array}
\]

4 + ? = 9

\[
\begin{array}{c}
1 & 2 & 3 \hline \\
4 & 5 & 6 \\
\end{array}
\]

\[
\begin{array}{c}
1 & 2 & 3 \hline \\
4 & 5 & 6 \\
\end{array}
\]

? + 4 = 7

4 + 3 = 7

\[
\begin{array}{c}
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\cdot \cdot \cdot \\
\end{array}
\]

4 + 4 = 8 so 4 + 3 is one less or 7
Design Goals

• Make room for Invention
• Attend to representation and strategy
• Explicitly make connections
• Organize strategies and inventions
• Attend to strategy choice
Explicitly Make Connections
Explicitly Make Connections

1. Chris’s group made 28 hats. Julia’s group made 44 hats. How many hats did both groups make altogether?

   **Julia’s Strategy:**
   
   Altogether we made 72 hats. I broke the numbers into tens and ones: 20 + 40 is 60, 8 and 4 is 12, 60 + 12 is 72. We made 72 hats.

   ![Julia's illustration]

   
   28 = 20 + 8
   + 44 = 40 + 4

   \[60 + 12 = 72 \text{ hats}\]

   ![Chris's illustration]

   I can think about it better if I make a number line in my head. I think about starting at 44, moving forward 30 and then back 2, since 28 is 2 less than 30. I can write it like this.

   **Chris’s Strategy:**

   ![Number line illustration]

   I start at 44 and then add on 30, going by tens: 54, 64, 74. Subtract 2 and it is 72. 72 hats.

2. A. How did Julia use tens and ones to add?
   B. How did Chris use tens and ones?
Finish It: Addition

1. Ming, Rosa, Chris, and Levi started solving $876 + 421$. They did not finish. Estimate the sum. Help each student finish the problem using the method they chose.

   $876 + 421$

   Estimate __________________________

A. Ming used a number line:

   Number sentence __________________________

B. Rosa used expanded form:

   $876 = 800 + _____ + _____ + 421 = 400 + _____ + _____$

   _____ + _____ + _____ = _____

C. Chris used all-partial:

   $876 + 421$
   $1200$
   $oxed{7}$

D. Levi used the compact method:

   $876$
   $+ 421$
   $1297$
Explicitly Make Connections

Help Confused Contessa fix her mistakes.

<table>
<thead>
<tr>
<th>Tell your partner what went wrong.</th>
<th>Estimate, then use the same method to solve it.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. $835 = 8 + 3 + 5$</td>
<td></td>
</tr>
<tr>
<td>$+ 697 = 6 + 9 + 7$</td>
<td></td>
</tr>
<tr>
<td>$14 + 12 + 12 = 38$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$835 = \underline{\quad}$</td>
</tr>
<tr>
<td></td>
<td>$+ 697 = \underline{\quad}$</td>
</tr>
</tbody>
</table>

Estimate $\underline{\quad}$
What strategy did this student use?

What do you know about this student from this snapshot?

One more person. How did you solve that problem: 62 + 30?
Strategy Session or Seminar

What strategy did this student use?

What do you know about this student from this snapshot?

Are all students in the same place?

[Adding: 79 + 28] I think um...I made 79 circles and 28 squares...
Design Goals

• Make room for Invention
• Attend to representation and strategy
• Explicitly make connections
• Organize strategies and inventions
• Attend to strategy choice
## My Addition Strategies Menu for Larger Numbers

<table>
<thead>
<tr>
<th>Counting All</th>
<th>Making Ten</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting On</td>
<td>Using Ten</td>
</tr>
<tr>
<td>Another Strategy</td>
<td>Using Doubles</td>
</tr>
</tbody>
</table>
Addition Strategies Menu for the Facts

### Counting All

10 + 2 = 12

### Making Ten

8 + 4 = 12

### Counting On

9 + 4 = 13

### Using Ten

9 + 8 = 17

### Using Doubles

6 + 7 = 13

Another Strategy ________________
**Addition Strategies Menu**

**Finding Friendly Numbers**
1. **138 + 29**
   - 140 + 30 = 170
   - 170 is a reasonable estimate.
   - Levi

**Using Base-Ten Pieces**
1. **68**
   + 55
   = 123
   - Peter
   - Trade 11 skinnies and 13 bits for 1 flat, 2 skinnies, and 3 bits

**Counting On**
1. **138 + 29**
   - 138 + 30 - 1 = 167
   - Yolanda

**Using Expanded Form**
1. **68 = 60 + 8**
   + 55 = 50 + 5
   = 110 + 13 = 123
   - Tera

**Using All-Partials**
1. **68**
   + 55
   = 110
   - 13
   = 123
   - Josh

**Using the Compact Method**
1. **167 + 68**
   + 55
   = 123
   - Julia
“Students with procedural deficits use less advanced methods than their peers. Although many eventually catch-up, this long period of using primitive methods may be detrimental.” (Fuson)
Use Menus to prompt. . .

Try a method you hardly ever choose.

Show Tanya’s method using a number line instead.

Which strategy do you think is best?

Carols got stuck. . . .what strategy do you think will help him?

Is your strategy similar to . . . .
Design Goals

- Make room for Invention
- Attend to representation and strategy
- Explicitly make connections
- Organize strategies and inventions
- Attend to strategy choice
What strategy would you use?

- Display the strategies on the pink sheets of paper.
- Find the baggie of addition facts.

Sort the facts by strategy.
There are 5204 Chocos. A customer came in and bought 565. Another customer came in and wanted to buy 4859 pieces of candy. Was there enough candy in the store so that he could buy that much?
Addition Strategies

Jan's Strategy
5 + 3 = 8

Bill's Strategy
4 + ? = 9

Jean's Strategy
? + 4 = 7

Sam's Strategy
4 + 1 = 5

Zero Strategy
2 + 0 = 2

Using Doubles
9 + 9 = 18 so 9 + 7 = 18 - 2

Using Ten
10 + 7 = 17

Making Ten
8 + 6 4 + 8

Making Ten
7 + 5 5 + 8

Counting On
8 + 3 9 + 2

Counting All
2 + 0

Drawing:

I have 2 and add no more so I have 2.
Stages of Conceptual Development

Direct Modeling

Counting Strategies

Reasoning from Known Facts

Estimation

Mental Math

Paper and Pencil

Carpenter, 1999; National Research Council, 2001
What do you notice about this student?

Student #2
Making Meaning from Multiple Strategies

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References


Representations of Number

- Quantity (counters)

- Number lines/ Number Path
Practices to consider for Students with Special Needs

• Provide instruction that explicitly helps connect models, representations, and concepts. (Fuson, 1992; Ginsburg, 1997)
• Provide (but guide) student choices of strategies (Gersten and Chard, 2001)
• Shorter, carefully constructed problem sets may be more effective in helping develop fluency in facts and procedures. (Diezman et al, 2003)
“Encouraging children to use efficient strategies to derive unknown facts before drill is better than ‘premature drill’ . . . and doing so increases both initial learning and retention.”

Focus on Strategic Thinking

<table>
<thead>
<tr>
<th>Group</th>
<th>Addition Facts</th>
<th>Strategies Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0 + 1, 1 + 1, 2 + 1, 3 + 1, 0 + 2, 2 + 2, 3 + 2, 4 + 2</td>
<td>Counting On, Zero</td>
</tr>
<tr>
<td>B</td>
<td>3 + 0, 4 + 0, 4 + 1, 5 + 1, 6 + 1, 5 + 2, 6 + 2, 5 + 3, 7 + 1, 3 + 1 + 8</td>
<td>Counting On, Zero</td>
</tr>
<tr>
<td>C</td>
<td>1 + 9, 2 + 7, 2 + 8, 2 + 9, 3 + 6, 3 + 7, 3 + 8, 4 + 6, 4 + 7, 5 + 5, 5 + 6</td>
<td>Making Ten, Using Ten</td>
</tr>
<tr>
<td>D</td>
<td>3 + 3, 3 + 4, 4 + 4, 4 + 5, 6 + 6, 6 + 7, 7 + 7, 8 + 8, 8 + 8, 10 + 9, 10 + 10</td>
<td>Using Doubles</td>
</tr>
<tr>
<td>E</td>
<td>5 + 7, 8 + 4, 8 + 5, 9 + 3, 9 + 4, 9 + 5, 10 + 1, 10 + 2, 10 + 3</td>
<td>Making Ten, Using Ten</td>
</tr>
<tr>
<td>F</td>
<td>8 + 6, 9 + 6, 9 + 7, 10 + 4, 10 + 5, 10 + 6, 10 + 7, 10 + 8, 9 + 8, 9 + 9</td>
<td>Making Ten, Using Ten</td>
</tr>
</tbody>
</table>
Focus on Strategic Thinking

<table>
<thead>
<tr>
<th>Group</th>
<th>Addition Facts</th>
<th>Strategies Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$0 + 1, 1 + 1, 1 + 2, 1 + 3, 1 + 0 + 2, 2 + 2, 2 + 3, 2 + 4 + 2$</td>
<td>Counting On, Zero</td>
</tr>
<tr>
<td>B</td>
<td>$3 + 0, 4 + 0, 4 + 1, 5 + 1, 6 + 1, 5 + 2, 6 + 2, 5 + 3, 7 + 1, 1 + 8$</td>
<td>Counting On, Zero</td>
</tr>
<tr>
<td>C</td>
<td>$1 + 9, 2 + 7, 2 + 8, 2 + 9, 3 + 6, 3 + 7, 3 + 8, 4 + 6, 4 + 7, 5 + 5, 5 + 6$</td>
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</tr>
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</tr>
<tr>
<td>F</td>
<td>$8 + 6, 9 + 6, 9 + 7, 10 + 4, 10 + 5, 10 + 6, 10 + 7, 10 + 8, 9 + 8, 9 + 9$</td>
<td>Making Ten, Using Ten</td>
</tr>
</tbody>
</table>

**Figure 4: Addition Facts Groups for Grade 2**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Subtraction Facts</th>
<th>Focus</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Groups A and B</td>
<td>Use strategies fluently for facts with sums to ten.</td>
</tr>
<tr>
<td>2</td>
<td>Group C</td>
<td>Develop mental math strategies and number sense and solve fact families for facts with sums more than ten.</td>
</tr>
<tr>
<td>3</td>
<td>Group D</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Group E</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Group F</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Groups C and D</td>
<td>Use strategies fluently and solve facts families</td>
</tr>
<tr>
<td>7</td>
<td>Group E</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Group F</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 5: Development of addition facts and the related subtraction facts in Grade 2**
• **Direct modeling** in which students re-create the action;

• **Counting strategies** such as counting on and counting back; and

• **Reasoning from known facts** in which students work from facts they already know. For example, if a student knows 5 + 5, then he or she has a quick way to access 4 + 5

Carpenter, 1999; National Research Council, 2001
Promote Problem Solving and Invention

- Cover Up
“Opportunities to construct their own procedures provide students with opportunities to make connections between the strands of proficiency. Procedural fluency is built directly on their understanding. The invention itself is a kind of problem solving, and they must use reasoning to justify their invented procedure. Students who have invented their own correct procedures also approach mathematics with confidence rather than fear and hesitation.”

Kamii and Dominick, 1998 and National Research Council, 2001, p. 197
Strategy Session or Seminar

What strategy did this student use?

What do you know about this student from this snapshot?